TH①. 수열의 합

STEP 1 Plains

1 수열의 합

① 시그마

(1) $a_1+a_2+a_3+\cdots+a_n=\sum_{k=1}^{n}a_k=S_n$

(2) $a_m+a_{m+1}+a_{m+2}+\cdots+a_n=\sum_{k=m}^{n}a_k=S_n-S_{m-1}$

(3) $\sum_{k=1}^{n}(a_{2k-1}+a_{2k})=\sum_{k=1}^{2n}a_k$

② 시그마의 기본 성질

(1) $\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$, $\sum_{k=1}^{n}(a_k-b_k)=\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}b_k$

(2) $\sum_{k=1}^{n}ca_k=c\sum_{k=1}^{n}a_k$

(3) $\sum_{k=1}^{n}c=cn$

(4) $\sum_{j=1}^{n}ij=i\sum_{j=1}^{n}j$

(5) $\sum_{k=1}^{n}a_kb_k\neq\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k$

(6) $\sum_{k=1}^{n}\frac{a_k}{b_k}\neq\frac{\sum_{k=1}^{n}a_k}{\sum_{k=1}^{n}b_k}$

① 자연수 거듭제곱의 합

(1) $\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

(2) $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\cdots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

(4) $\sum_{k=1}^{n} (2k-1) = n^2$ **(홀수의 합)**

(5) $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

(6) $\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

② 등차/등비수열과 시그마

(1) $a_n = an+b$ **(일반항이 일차식 ➜ 등차수열)**

- $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (ak+b) = \frac{n(a+(an+b))}{2}$
- $\sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (ak+b) = \frac{(n-m+1)((am+b)+(an+b))}{2}$

(2) $a_n = a(b)^n$ **(일반항이 지수를 포함한 식 ➜ 등비수열)**

- $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a(b)^k = \frac{ab(b^n-1)}{b-1}$
- $\sum_{k=m}^{n} a_k = \sum_{k=m}^{n} a(b)^k = \frac{ab^m(b^{n-m+1}-1)}{b-1}$

③ 다양한 등차수열의 합

(1) **'등차수열의 합 = $\frac{(\text{항의 개수})(\text{첫항}+\text{마지막항})}{2}$'을 활용해보자.**

- $\sum_{k=1}^{n} (a_k + a_{k+p}) = \frac{n((a_1+a_{1+p})+(a_n+a_{n+p}))}{2}$

- $\sum_{k=1}^{n} a_{pk+q} = \frac{n(a_{p+q}+a_{pn+q})}{2}$

- $\sum_{k=1}^{n} (a_k+b_k) = \frac{n((a_1+b_1)+(a_n+b_n))}{2}$

④ 다양한 등비수열의 합

(1) ‘등비수열의 합 $= \dfrac{\text{첫째항} \times (\text{공비}^{\text{항의 개수}} - 1)}{\text{공비} - 1}$’을 활용해보자.

- $\displaystyle\sum_{k=1}^{n} ar^{bk+c} = \frac{ar^{b+c}((r^b)^n - 1)}{r^b - 1}$

- $\displaystyle\sum_{k=1}^{n} a_{pk+q} = \frac{a_{p+q}((r^p)^n - 1)}{r^p - 1}$

- $\displaystyle\sum_{k=1}^{n} (a_k)^p = \frac{(a_1)^p((r^p)^n - 1)}{r^p - 1}$

- $\displaystyle\sum_{k=1}^{n} \left(\frac{1}{a_k}\right)^p = \frac{\left(\frac{1}{a_1}\right)^p\left(1 - \left(\left(\frac{1}{r}\right)^p\right)^n\right)}{1 - \left(\frac{1}{r}\right)^p}$

- $\displaystyle\sum_{k=1}^{n} (a_k + a_{k+p}) = \frac{(a_1 + a_{1+p})(r^n - 1)}{r - 1}$

- $\displaystyle\sum_{k=1}^{n} a_k b_k = \frac{a_1 b_1((r_a r_b)^n - 1)}{r_a r_b - 1}$

- $\displaystyle\sum_{k=1}^{n} \frac{a_k}{b_k} = \frac{\frac{a_1}{b_1}\left(1 - \left(\frac{r_a}{r_b}\right)^n\right)}{1 - \frac{r_a}{r_b}}$

⑤ 교대수열의 합

(1) $\displaystyle\sum_{k=1}^{n} (a_{k+1} - a_k) = a_{n+1} - a_1$

(2) $\displaystyle\sum_{k=1}^{n} (a_{k+2} - a_k) = a_{n+2} + a_{n+1} - a_2 - a_1$

(3) $\displaystyle\sum_{k=1}^{n} (-1)^k(a_{k+1} + a_k) = -a_1 + (-1)^n a_{n+1}$

⑥ 분수 꼴의 교대수열의 합

(1) $\sum_{k=1}^{n}\frac{a_{k+1}-a_k}{a_ka_{k+1}}=\sum_{k=1}^{n}\left(\frac{1}{a_k}-\frac{1}{a_{k+1}}\right)=\frac{1}{a_1}-\frac{1}{a_{n+1}}$

(2) $\sum_{k=1}^{n}\frac{a_{k+2}-a_k}{a_ka_{k+2}}=\sum_{k=1}^{n}\left(\frac{1}{a_k}-\frac{1}{a_{k+2}}\right)=\frac{1}{a_1}+\frac{1}{a_2}-\frac{1}{a_{n+1}}-\frac{1}{a_{n+2}}$

(3) $\sum_{k=1}^{n}\frac{(-1)^k(a_k+a_{k+1})}{a_ka_{k+1}}=\sum_{k=1}^{n}(-1)^k\left(\frac{1}{a_k}+\frac{1}{a_{k+1}}\right)=-\frac{1}{a_1}+\frac{(-1)^{n+1}}{a_{n+1}}$

(4) $\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\frac{n}{n+1}$

(5) $\sum_{k=1}^{n}\frac{1}{k(k+2)}=\sum_{k=1}^{n}\frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+2}\right)=\frac{1}{2}\left\{\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{n+1}+\frac{1}{n+2}\right)\right\}=\frac{n(3n+5)}{4(n+1)(n+2)}$

(6) $\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\sum_{k=1}^{n}\frac{1}{2}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{2n+1}\right)=\frac{n}{2n+1}$

(7) $\sum_{k=1}^{n}\frac{1}{(k+1)(k+2)}=\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=\frac{1}{2}-\frac{1}{n+2}=\frac{n}{2n+4}$

(8) $\sum_{k=1}^{n}\frac{(-1)^k(2k+1)}{k(k+1)}=\sum_{k=1}^{n}\frac{(-1)^k\{k+(k+1)\}}{k(k+1)}=\sum_{k=1}^{n}(-1)^k\left(\frac{1}{k}+\frac{1}{k+1}\right)=(-1)+\frac{(-1)^n}{n+1}$

(9) a_n이 등차수열일 때

- $\sum_{k=1}^{n}\frac{1}{a_ka_{k+1}}=\sum_{k=1}^{n}\frac{1}{d}\left(\frac{1}{a_k}-\frac{1}{a_{k+1}}\right)=\frac{1}{d}\left(\frac{1}{a_1}-\frac{1}{a_{n+1}}\right)$
- $\sum_{k=1}^{n}\frac{1}{a_ka_{k+2}}=\sum_{k=1}^{n}\frac{1}{2d}\left(\frac{1}{a_k}-\frac{1}{a_{k+2}}\right)=\frac{1}{2d}\left(\frac{1}{a_1}+\frac{1}{a_2}-\frac{1}{a_{n+1}}-\frac{1}{a_{n+2}}\right)$

(10) S_n이 수열의 합일 때

- $\sum_{k=1}^{n}\frac{a_{k+1}}{S_kS_{k+1}}=\sum_{k=1}^{n}\frac{a_{k+1}}{a_{k+1}}\left(\frac{1}{S_k}-\frac{1}{S_{k+1}}\right)=\left(\frac{1}{S_1}-\frac{1}{S_{n+1}}\right)=\left(\frac{1}{a_1}-\frac{1}{S_{n+1}}\right)$

⑦ 일반항의 분모가 유리식인 수열의 합

(1) $\sum_{k=1}^{n}\frac{1}{\sqrt{k+2}+\sqrt{k+1}}=\sum_{k=1}^{n}\frac{\sqrt{k+2}-\sqrt{k+1}}{(\sqrt{k+2}+\sqrt{k+1})(\sqrt{k+2}-\sqrt{k+1})}=\sum_{k=1}^{n}(\sqrt{k+2}-\sqrt{k+1})=\sqrt{n+2}-\sqrt{2}$

(2) $\sum_{k-1}^{n}\frac{1}{\sqrt{k+2}+\sqrt{k}}=\sum_{k=1}^{n}\frac{\sqrt{k+2}-\sqrt{k}}{(\sqrt{k+2}+\sqrt{k})(\sqrt{k+2}-\sqrt{k})}=\frac{1}{2}\sum_{k=1}^{n}(\sqrt{k+2}-\sqrt{k})=\frac{1}{2}(-1-\sqrt{2}+\sqrt{n+1}+\sqrt{n+2})$

⑧ 일반항이 로그의 형태로 주어진 수열의 합

(1) $\sum_{k=1}^{n}\log\left(\frac{1}{k}+1\right)=\sum_{k=1}^{n}\log\left(\frac{k+1}{k}\right)=\log(n+1)$

(2) $\sum_{k=2}^{n}\log\left(1-\frac{1}{k^2}\right)=\sum_{k=2}^{n}\log\left(\frac{k-1}{k}\cdot\frac{k+1}{k}\right)=\log\left(\frac{1}{2}\cdot\frac{n+1}{n}\right)$

3 수열의 합과 일반항의 관계

① $\sum_{k=1}^{n} a_k = f(n)$일 때, a_n을 구하기

$\sum_{k=1}^{n} a_k = f(n) = S_n$

$- \quad \sum_{k=1}^{n-1} a_k = f(n-1) = S_{n-1}$

$a_n = f(n) - f(n-1)$ (단, $n \geq 2$)

- **$\sum_{k=1}^{n} a_k = f(n)$이 식에 $n=1$을 대입하여 나온 a_1과 $a_n = f(n) - f(n-1)$에 $n=1$을 대입하여 나온 a_1이 일치한다면**

  - $a_n = f(n) - f(n-1)$ $(n \geq 1)$

- **$\sum_{k=1}^{n} a_k = f(n)$이 식에 $n=1$을 대입하여 나온 a_1과 $a_n = f(n) - f(n-1)$에 $n=1$을 대입하여 나온 a_1이 일치하지 않는다면**

  - $\sum_{k=1}^{n} a_k = f(n)$에 $n=1$을 대입하여 나온 a_1이 진짜 a_1
  - $a_n = f(n) - f(n-1)$ $(n \geq 2)$